

# A HIERARCHY OF GAUGED GRASSMANIAN MODELS IN $4p$ DIMENSIONS WITH SELF-DUAL INSTANTONS

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## Abstract

We present a hierarchy of gauged Grassmanian models in  $4p$  dimensions, where the gauge field takes its values in the  $2^{2p-1} \times 2^{2p-1}$  chiral representation of  $SO(4p)$ . The actions of all these models are absolutely minimised by a hierarchy of self-duality equations, all of which reduce to a single pair of coupled ordinary differential equations when subjected to  $4p$  dimensional spherical symmetry.

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# 1 Gauging Sigma Models

In addition to introducing covariant derivatives in terms of the gauge connection, and curvature dependent terms that describe the dynamics of the gauge field, our prescription for gauging a Sigma model includes a crucial additional requirement: the action/energy density of the gauged Sigma model must be bounded from below by a total divergence whose (surface) integral can be arranged to take on a non vanishing value by requiring suitable asymptotic conditions for the classical solutions. These are the topologically stable instantons/solitons with finite action/energy.

The gauging of Sigma models in 3 dimensions has been considered by Witten[1] and Rubakov[2] for the Skyrme model[3], which is equivalent to the  $O(4)$  Sigma model. In both these cases there is no topological inequality bounding the energy from below, and indeed the lack of (topological) stability has been exploited[2] in the context of Techniskyrmions.

By contrast, the gauging of the  $O(3)$  Sigma model in 2 dimensions was considered by Mehta, Davis and Aitchison[4] in the context of a 2+1 dimensional Chern-Simons theory, where finite energy vortex solutions were found. The topological inequalities that are responsible for the stability of these vortices were given in Ref[5] and subsequently this analysis was extended[6] to an  $O(3)$  model augmented by a Skyrme term. The topological inequalities employed in Refs.[4],[5],[6] were stated not for the  $O(3)$  model but for the classically equivalent  $CP^1$  model which happens to be a Grassmanian model. In Ref.[6], it was described in some detail how the gauging of the  $O(3)$  model with a  $U(1)$  gauge field did not lead to a non-trivial topological lower bound on the energy while gauging the equivalent  $CP^1$  model did just that.

Motivated by the above, instead of gauging  $2n$  dimensional  $O(2n + 1)$  Sigma models for which we do not expect to find topological inequalities, we consider the gauging of a certain hierarchy of Grassmanian models in  $2n$  dimensions which we shall show to be endowed with the requisite topological inequalities implying stability and lower bounds on their actions. These  $2n$  dimensional Grassmanian models will be gauged with a connection field taking its values in the chiral representation of  $SO(2n)$ . If we allowed the integer

$n$  to be both even and odd, then this would have been a direct generalisation of the  $SO(2) \approx U(1)$  gauged 2 dimensional  $CP^1$  model of Refs.[4],[5],[6], where the  $U(1)$  gauged  $CP^1$  model would have constituted the first member of the  $2n$  dimensional hierarchy of  $SO(2n)$  gauged Grassmannian models. In this paper we restrict ourselves to only even values of the integer  $n$  because we intend to restrict ourselves to a hierarchy of scale invariant systems. A more general classification including systems that are not scale invariant will be undertaken elsewhere. As a consequence of their scale invariance the Euler-Lagrange equations of this hierarchy of models are solved by a hierarchy of first order self-duality equations. The self-duality equations are presented, and this is followed by some discussion of the physical relevance of the 4 dimensional case.

## 2 The Gauged Grassmanian Models

The Grassmanian models in  $2n$  dimensions that we consider are described by the fields

$$z^A_i = (z^a_i, z^\alpha_i), \quad i, a, \alpha = 1, \dots, 2^{n-1}; \quad (1)$$

subject to the constraint

$$(z^*)^i_A z^A_j = \delta^i_j \quad (2)$$

or more briefly  $z^\dagger z = 1$ . The covariant derivative of the field  $z$  is defined by

$$D_\mu z = \partial_\mu z - z A_\mu \quad (3)$$

where the connection  $A_\mu$  takes its values in the  $2^{n-1} \times 2^{n-1}$  dimensional chiral representation of  $SO(2n)$ , and is not here the composite field  $A_\mu = z^\dagger \partial_\mu z$  of a pure Grassmanian model.

The action of the local gauge group element  $g$  is given by

$$z \rightarrow zg \quad (4)$$

$$D_\mu z \rightarrow D_\mu zg \quad (5)$$

Let us henceforth restrict to even values of  $n = 2p$  and define the following three tensor field strengths  $F(2p)$ ,  $G(2)$  and  $H(2p)$  as follows: The  $2p$  form field  $F(2p)$  is the  $p$  fold totally antisymmetrised product of the curvature 2 form  $F(2) \equiv F_{\mu\nu}$

$$F(2p) \equiv F_{\mu_1\mu_2\ldots\mu_{2p}} = F_{[\mu_1\mu_2\ldots\mu_{2p-1}\mu_{2p}]} \quad (6)$$

The square brackets on the indices  $\mu_1, \mu_2, \ldots$  imply total antisymmetrisation. The 2 form field  $G(2)$  is defined by

$$G(2) \equiv G_{\mu\nu} = D_{[\mu} z^\dagger D_{\nu]} z \quad (7)$$

The  $2p$  form field  $H(2p) = G(2) \wedge F(2p-2)$  is then given by

$$H(2p) \equiv H_{\mu_1\mu_2\ldots\mu_{2p}} = G_{[\mu_1\mu_2} F_{\mu_3\ldots\mu_{2p}]} \quad (8)$$

Note that  $G(2)$  and hence also  $H(2p)$  are both gauge covariant quantities as seen from (5), since  $F(2p)$  is by definition gauge covariant. The curvature 2 form here is defined by  $D_{[\mu} D_{\nu]} z = -z F_{\mu\nu}$ . Note also that  $H(2p)$ , which depends on the curvature form  $F(2p-2)$  is independent of the curvature 2 form  $F(2)$  in 4 dimensions only, in which case  $H(2) = G(2) \wedge F(0) \equiv G(2)$ .

Using the  $F$  and  $H$  field strengths, we proceed to state the inequalities which will give rise to topological stability, and at the same time define the action densities which are bounded from below by the respective topological charge densities. These inequalities are

$$Tr[\star F(2p) \mp H(2p)]^2 \geq 0 \quad (9)$$

where the notation  $\star F$  implies the definition for the Hodge dual of  $F$ . The inequality (9) yields both the definition of the action density and the topological inequality

$$\mathcal{L}_p \stackrel{\text{def}}{=} Tr[F(2p)^2 + H(2p)^2] \geq \pm 2Tr\star F(2p)H(2p) \quad (10)$$

In (9) and (10), the trace is taken over the indices of the local gauge group, c.f. (5), and hence all quantities are gauge invariant.

It is easy to verify that the right hand side of (10) is a total divergence. This is seen if we reexpress this density as  $Tr\star F(4p-2)G(2)$ , and make use

of the Bianchi identities as well as the constraint (2). The topological charge density  $\varrho$  is then

$$\begin{aligned}\varrho &= Tr(\star F(4p-2))_{\mu\nu} G_{\mu\nu} \\ &\simeq \varepsilon_{\mu_1\mu_2\mu_3\mu_4\ldots\mu_{4p}} [Tr F_{\mu_1\mu_2} F_{\mu_3\mu_4} \cdots F_{\mu_{4p-1}\mu_{4p}} + 2\partial_{\mu_1} Tr F_{\mu_3\mu_4} \cdots F_{\mu_{4p-1}\mu_{4p}} z^\dagger D_{\mu_2} z] \\ &\quad (11)\end{aligned}$$

Thus the hierarchy of gauged Grassmanian models in every  $4p$  dimension is characterised by the Lagrangian density  $\mathcal{L}_p$  on the left hand side of (10), which is bounded from below by the total divergence (11). We notice that the first term in (11) is the  $2p$ -th Chern-Pontryagin(C-P) density. This quantity is the total divergence of the Chern-Simons(C-S) density while the second term is already in total divergence form. That solutions exist for which these surface integrals are finite and nonvanishing will be shown below when we restrict to the spherically symmetric field configurations.

We notice here that the  $4p$  dimensional action integral of the Lagrange density  $\mathcal{L}_p$  given by the left hand side of (10) is scale invariant. This is because the 2 form field  $G_{\mu\nu}$  defined in (7) scales exactly like the curvature 2 form  $F_{\mu\nu}$  since the Grassman-valued fields  $z$  are themselves dimensionless. This is reflected by the absence of a dimensional constant in the inequality (9), which means that the self duality equations which saturate (9)

$$\star F(2p) = \pm H(2p) \quad (12)$$

also do not feature a dimensional constant.

The close analogy with the  $4p$  dimensional Yang-Mills hierarchy[7] [8] suggests that the relevant solutions of (12) will have a pure-gauge behaviour at infinity. For the latter statement, it is necessary to first choose the gauge group, and since equations (12) are not restricted to take values inside the algebra of this gauge group, we must specify further the representation in which the connection field is defined. This was stated above as being the chiral representation of  $SO(4p)$ , for the  $4p$  dimensional member of this hierarchy.

### 3 Spherically Symmetric Fields

To explain the problem of imposing spherical symmetry, we find it most convenient to employ the formalism of Schwarts[9]. Very briefly, this amounts to solving an algebraic constraint for each of the two fields involved, namely the gauge connection  $A_\mu = (A_i, A_{4p})$ ,  $i = 1, \dots, 4p - 1$  and the Grassmanian field  $z$ . These equations stated at the north pole of the  $4p$ -sphere, are, for the gauge connection

$$A_0 = 0 \tag{13}$$

$$\mu^{-1} A_i \mu = h_i^j A_j \tag{14}$$

where  $h$  is an element of  $SO(4p - 1)$ , the stability group of the north pole, and  $\mu$  is the appropriate representation of  $h$  which in our case is the chiral representation of  $SO(4p)$  in terms of gamma matrices. For the Grassmanian field the spherical symmetry constraint is

$$z\mu = z \tag{15}$$

While equation(14) can be solved readily as was done in Ref.[8] and elsewhere, equation(15) cannot be solved except for the case  $n = 1$  in equation(1), namely the case of the  $CP^1$  model with  $U(1)$  gauge freedom in 2 dimensions which was considered in Ref[4]. This is not surprising since in 2 dimensions the  $CP^1$  model is equivalent to the  $O(3)$  model, and the symmetry equation corresponding to (15) for all  $d$  dimensional  $O(d + 1)$  models[10] is solvable. This means that any radially symmetric Ansatz that we make here will not be strictly spherically symmetric and hence due care must be taken for its consistency. This involves the consistency of the Euler-Lagrange equations arising from the variation of the one dimensional subsystem resulting from the application of the Ansatz, with the Euler-Lagrange equations of the system obtained before the application of the said Ansatz.

In the present paper, the question of this consistency of the Ansatz does not arise because we shall not be applying the variational principle to the reduced one dimensional subsystem, but instead will be solving the full self duality equations. In future applications however, when we would envisage extending the present models (10) by higher/lower order terms in the derivatives of the fields, the solutions would be necessarily non-self dual and hence it would be necessary to verify the consistency of the Ansatz.

Our Ansatz for the connection  $A_\mu$  satisfies the symmetry restriction (14) and is therefore strictly spherically symmetric. Using  $r$  as the radial variable and  $\hat{x}_\mu$  as the unit vector in  $4p$  dimensions, this is

$$A_\mu = \frac{2}{r}[1 - k(r)]\tilde{\Sigma}_{\mu\nu}\hat{x}_\nu \quad (16)$$

The tensor valued matrices  $\tilde{\Sigma}_{\mu\nu}$  in (16), and their chiral conjugates  $\Sigma_{\mu\nu}$  are defined by

$$\tilde{\Sigma}_{\mu\nu} = -\frac{1}{4}\tilde{\Sigma}_{[\mu}\Sigma_{\nu]}, \quad \Sigma_{\mu\nu} = -\frac{1}{4}\Sigma_{[\mu}\tilde{\Sigma}_{\nu]} \quad (17)$$

in terms of the vector valued matrices  $\tilde{\Sigma}_\mu$  and  $\Sigma_\mu$  are in turn defined in terms of the gamma matrices  $\Gamma_\mu$  in  $4p$  dimensions and the respective chiral matrix  $\Gamma_{4p+1}$  as

$$\tilde{\Sigma}_\mu = \frac{1}{2}(1 - \Gamma_{4p+1})\Gamma_\mu, \quad \Sigma_\mu = \frac{1}{2}(1 + \Gamma_{4p+1})\Gamma_\mu \quad (18)$$

Our Ansatz for the Grassmanian valued field  $z^A_i = (z^a_i, z^\alpha_i)$  is

$$z^a_i = \sin\frac{f(r)}{2}\delta^a_i, \quad z^\alpha_i = \cos\frac{f(r)}{2}\hat{x}_\mu(\Sigma_\mu)^\alpha_i \quad (19)$$

It is now a straightforward matter to compute the curvature 2 form  $F_{\mu\nu}$  and the covariant derivative  $D_\mu z$  for the field configuration given by (16) and (19) and hence to evaluate the  $2p$  form fields  $F(2p)$  and  $H(2p)$ , with a view to substituting the latter in the self duality equations(12). To analyse the resulting equations in the functions  $k(r)$  and  $f(r)$  we shall need the tensor-spinor identities employed in Ref.[8]

$$\Sigma(2p) = \star \Sigma(2p), \quad \tilde{\Sigma}(2p) = -\star \tilde{\Sigma}(2p) \quad (20)$$

where we have used the following notation for  $\Sigma(2p) \equiv \Sigma_{\mu_1\mu_2\dots\mu_{2p}}$  and  $\tilde{\Sigma}(2p) \equiv \tilde{\Sigma}_{\mu_1\mu_2\dots\mu_{2p}}$ , analogous with (6),

$$\Sigma_{\mu_1\mu_2\dots\mu_{2p}} = \Sigma_{[\mu_1}\tilde{\Sigma}_{\mu_2}\dots\Sigma_{\mu_{2p-1}}\tilde{\Sigma}_{\mu_{2p}]}, \quad \tilde{\Sigma}_{\mu_1\mu_2\dots\mu_{2p}} = \tilde{\Sigma}_{[\mu_1}\Sigma_{\mu_2}\dots\tilde{\Sigma}_{\mu_{2p-1}}\Sigma_{\mu_{2p}]} \quad (21)$$

The resulting first order differential equations for the functions  $k(r)$  and  $f(r)$  are the same for all  $p$ , namely for all members of the hierarchy. This is exactly

the same situation as for the  $4p$  dimensional hierarchy of scale invariant Yang-Mills[7] models. In that case(??) the corresponding hierarchy of self duality equations reduced to a single first order equation, which yields the well known BPST instanton[11].

In the present case, the these coupled first order equations are, say in the self dual case,

$$k' = \frac{2}{r}[k^2 + \frac{1}{2}(2k-1)(\cos f - 1)], \quad f' \sin f = -\frac{1}{r}k(k-1) \quad (22)$$

It should be noted here that the radial variable  $r$  in (22) can be replaced with the dimensionless variable  $\rho$ , where  $\rho = \lambda r$  in terms of an arbitrary scale  $\lambda$ . As a result, these instantons will be localised to an arbitrary scale, which is a consequence of the scale invariance of  $\mathcal{L}_p$ .

The asymptotic behaviour of the solutions will be found in the next section where we examine the 4 dimensional  $p = 1$  case in detail. We shall see there, that the asymptotics found for the  $p = 1$  case hold also for all  $p$ . We shall therefore anticipate this large  $r$  behaviour and state that for arbitrary  $p$  the solutions of (22) will be expressed in terms of the matrix  $\omega = \hat{x}_\mu \tilde{\Sigma}_\mu$  in the pure-gauge form

$$A_\mu = \omega^{-1} \partial_\mu \omega, \quad z^a_i = 0, \quad z^\alpha_i = \omega^\alpha_i \quad (23)$$

which leads to the usual instanton charge when substituted in the topological charge density  $\rho$  given by (11). (23) is a consequence of the scale invariance of the hierarchy of gauged Grassmanian models characterised by the Lagrangians  $\mathcal{L}_p$  given in (10). The pure-gauge behaviour (23) can also be understood when we note that otherwise, if  $F_{\mu\nu}$  decayed as the inverse square of the radius as it does in the hierarchy of  $SO(d)$  Higgs models in  $d$  dimensions[12], the square of the  $2p$  form field  $F(2p)$  in  $\mathcal{L}_p$  here would cause the logarithmic divergence of the action.

Before proceeding to the details of  $p = 1$ , let us note two features of the general  $p \geq 1$  case. The first is that only in the  $p = 1$  case is the matrix  $\omega$  an element of the full gauge group, chiral  $SO(4) \equiv SO(4)_\pm$ , namely  $SU(2)$ . In all other cases with  $p > 1$ ,  $\omega$  is parametrised by the  $4p - 1$



independent components of  $\hat{x}_\mu$  which is successively smaller than the number of parameters of  $SO(4p)$ , with increasing  $p$ . The second feature is that the self duality equations (12) are successively more overdetermined for increasing  $p > 1$ . This is the case also for the Yang-Mills hierarchy of self duality equations[13], in which case for  $p > 1$  the least symmetric solution is the axially symmetric instanton[14]. Here counting the number of equations in (12) is equal to  $\frac{(4p)!}{(2p)!^2}$ . This number increases with  $p$  while the number of fields does not. Clearly the imposition of full spherical symmetry resulted in equations (22) which are not overdetermined. The question as to the existence of axially symmetric or other solutions belongs to the subject of future investigations.

## 4 $p=1$ : Four Dimensions

We consider this case on its own in more detail not only because this is the physically interesting case, but also as the simplest member of the hierarchy it has some special properties. The Lagrange density of the  $p = 1$  member of the hierarchy is

$$\mathcal{L}_1 = Tr[F_{\mu\nu}^2 + (D_{[\mu} z^\dagger D_{\nu]})^2] \quad (24)$$

and the topological charge density is

$$\varrho = 4\varepsilon_{\mu\nu\rho\sigma} Tr F_{\mu\nu} D_\mu z^\dagger D_\nu z \quad (25)$$

Substituting the field configuration given by (16) and (19) into (24) and (25) and integrating over the angular volume, we find the one dimensional effective Lagrange density  $L$  and topological charge density  $\sigma$  descending from (24) and (25) respectively to be

$$L = rk'^2 + \frac{4}{r}k^2(k-1)^2 + rf'^2 \sin^2 f + \frac{4}{r}[k^2 + \frac{1}{2}(2k-1)(\cos f - 1)]^2 \quad (26)$$

$$\sigma = \frac{d}{dr}[\frac{1}{3}k^3 + \frac{1}{2}k(k-1)(\cos f - 1)] \quad (27)$$

Note that the charge density  $\sigma$  for the spherically symmetric field configuration is a total derivative (27).

By inspecting the Lagrangian  $L$  (26) carefully, we conclude that there is only one set of asymptotic values for the functions  $k(r)$  and  $f(r)$  which is consistent with convergent action integral and non singular behaviour at the origin of  $r$ . This is

$$\lim_{r \rightarrow 0} k(r) = 1 \qquad \lim_{r \rightarrow \infty} k(r) = 0 \qquad (28)$$

$$\lim_{r \rightarrow 0} f(r) = \pi \qquad \lim_{r \rightarrow \infty} f(r) = 0 \qquad (29)$$

The asymptotic values (28) and (29) were deduced above by examining the reduced Lagrange density (26) for  $p = 1$ . It is not hard however to verify that (28) and (29) remain valid in the  $p > 1$  case as well. This justifies our anticipation of the pure-gauge behaviour (23) above, which is simply a consequence of the second member of (28).

Another interesting feature of the our  $4p$  dimensional instantons can be learnt from the  $p = 1$  case. This is the dependence of the topological charge on the small  $r$  behaviour of the function  $f(r)$  parametrising the Grassmanian field  $z$ . One might expect that the charge of our spherically symmetric instanton, regarded as the Hedgehog of a Grassmanian Sigma- Skyrme model, should depend on the small  $r$  behaviour. In the first member of (29)  $\pi$  can obviously be replaced by  $N\pi$  for any integer  $N$ . When this is done for the usual 3 dimensional Skyrme model [3] it is found that its topological charge (Baryon number) takes the value  $N$  and its energy becomes 2.98 times the energy of the unit charge Hedgehog[15]. It is interesting to see whether changing  $\pi$  to  $N\pi$  in the first member of (29) results in any change of the topological charge? The answer follows immediately from an inspection of (27). It is clear that the integral of (27) between the limits given by (28) and (29) is insensitive to this change and hence that our spherically symmetric instantons have a unique topological charge of one unit.

Interpreting the instanton of the model (25) as the vacuum[16] is quite natural since it behaves as a pure gauge at large  $r$ . On the other hand because of the scale invariance of(25) the instanton is localised to an arbitrary scale. If we wish to have an instanton localised to an absolute scale, we would have to add to the Lagrangian (24) several lower/higher order terms in the derivatives of the fields. These would be the quadratic kinetic term  $Tr D_\mu z^\dagger D_\mu z$  and

possibly a potential term. Both of these scale with a smaller power than (24) does, and hence to satisfy Derrick's theorem a higher order Skyrme term such as a sextic or an octic term in the derivatives must also be added to (24). This is a task of some complexity and is deferred to a future investigation.

Having mentioned the question of the vacuum of the theory being described by the instanton of the model, we end with the corresponding question pertaining to the finite energy solutions in the static limit. This is the question analogous to the one dealt with by Rubakov[2] in the case of the  $SU(2)$  gauged Skyrme model. As in Ref.[2] we would not expect to find a stable finite energy "soliton" solution. Indeed, the simplest way to demonstrate this is to proceed in the usual way by calculating the static Hamiltonian of the model in the temporal gauge and studying the finite energy solutions. Because of the multiplet structure of the Grassmanian field  $z$ , it is not possible to subject this field to spherical symmetry in 3 dimensions. This is completely analogous to the case in the Weinberg-Salam model[17] and the  $SO(4) \times U(1)$  Higgs model[18] where the solution is the unstable sphaleron characterised by the non contractible loop(NCL) field configuration for the field  $z^A_i = (z^a_i, z^\alpha_i)$

$$z^a_i = \cos \frac{\tilde{f}}{2} \delta^a_i, \quad z^\alpha_i = \sin \frac{\tilde{f}}{2} p_\mu(\mu, \theta, \phi) (\Sigma_\mu)^\alpha_i \quad (30)$$

where  $\Sigma_\mu = \tau_\mu = (\tau_i, \tau_4) = (i\sigma_i, 1)$  in the familiar  $p = 1$  case and  $\tilde{f}(\tilde{r})$  is a function of the radial variable  $\tilde{r}$  in 3 dimensions, and the 4 vector  $p_\mu(\mu, \theta, \phi)$  is

$$p_\mu = (\sin\mu \sin\theta \cos\phi, \sin\mu \sin\theta \sin\phi, \sin^2\mu \cos\theta + \cos^2\mu, \sin\mu \cos\mu (\cos\theta - 1)) \quad (31)$$

in which  $\theta$  and  $\phi$  are the polar and azimuthal angles in 3 dimensions while  $\mu$  is a constant parametrising the NCL. The sphaleron solution corresponds to the value  $\mu = \frac{\pi}{2}$  and is unstable against the variation of this parameter. A complete study of the sphaleron solutions of this model will be given in detail elsewhere.

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